Transport Properties of Random Walks on Scale-Free/Regular-Lattice Hybrid Networks

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We study numerically the mean access times for random walks on hybrid disordered structures formed by embedding scale-free networks into regular lattices, considering different transition rates for steps across lattice bonds (F) and across network shortcuts (f). For fast shortcuts ($f/F \gg 1$) and low shortcut densities, traversal time data collapse onto an universal curve, while a crossover behavior that can be related to the percolation threshold of the scale-free network component is identified at higher shortcut densities, in analogy to similar observations reported recently in Newman-Watts small-world networks. Furthermore, we observe that random walk traversal times are larger for networks with a higher degree of inhomogeneity in their shortcut distribution, and we discuss access time distributions as functions of the initial and final node degrees. These findings are relevant, in particular, when considering the optimization of existing information networks by the addition of a small number of fast shortcut connections.

I. INTRODUCTION

Empirical observations performed recently on real networks as different as the Internet and the World Wide Web, ecological and food webs, power grids and electronic circuits, genome and metabolic reactions, collaboration among scientists and among Hollywood actors, and many others, have shown some striking similarities in their structure and topology [1, 2]. In particular, the observations have revealed that the degree distributions of these networks are fat-tailed and typically close to a power-law, $P(k) \sim k^{-\gamma}$, where the exponent is usually in the range $2 < \gamma < 3$ [2, 3].

These findings, exciting because of their near-universal occurrence, have motivated numerous investigations on the topology and geometrical properties of scale-free (SF) and other types of complex networks, as well as many studies on dynamical and critical phenomena of statistical systems defined on complex network structures. Some of these efforts have been devoted towards understanding the transport properties of discrete time and continuous time random walks defined on different kinds of complex networks [4, 5, 6, 7, 8, 9, 10, 11, 12].

In particular, random walks have been studied on small-world networks (SWNs), which are structures formed by superimposing a classical random graph component (formed by randomly distributed long-range links or shortcuts) on a regular lattice [13, 14]. Among the recent contributions to this topic, Parris and Kenkre [11] introduced the important new feature of considering different jump rates for steps across regular lattice bonds (F) and across network shortcuts (f). This type of model is relevant to attempts to design, modify or optimize existing information networks in order to reduce the mean access time, by incorporating a small number of fast connections into an existing network already possessing a large number of other, perhaps slower, connections.

Ref. [11] focused on the *traversal time*, defined as the mean first-passage time for a random walker to reach the site furthest from its starting point, as measured along the 1D lattice (ring) backbone, averaged over different initial sites and network realizations. In this earlier study, a collapse of traversal times was found, with interesting universal behavior for $f/F \gg 1$ and low shortcut densities. By the term shortcut density n_{sw} we mean the ratio of the number of shortcut bonds in the system to the total number of sites in the network, i.e., one-half the average shortcut degree $\bar{k}$. In addition to the collapse, a crossover was observed at a critical density of shortcuts $n_{sw} = \bar{k}/2 \approx 1$, which was attributed to percolation of the random graph component of the network [11].

The aim of the present work is to extend the scope of these previous investigations by exploring random walks on hybrid network structures formed by superimposing SF networks onto regular lattices. We find that the data collapse and universal behavior previously observed with fast but low shortcut densities on SWNs are also observed in this type of hybrid structure, and that, consistent with the conclusions of Ref. [11], the crossover behavior associated with shortcut percolation is generally present in networks satisfying the Molloy-Reed criterion [15, 16]; such a transition is not seen in networks lacking a giant component. Furthermore, we find that an inhomogeneous distribution of links of the type that occurs in SF networks leads to generally larger traversal times when compared to networks with links distributed homogeneously, a result that bears resemblance to previous findings [17, 18, 19, 20] on the effects of network heterogeneity on two point resistances and synchronization efficiencies. Finally, we extend our study from
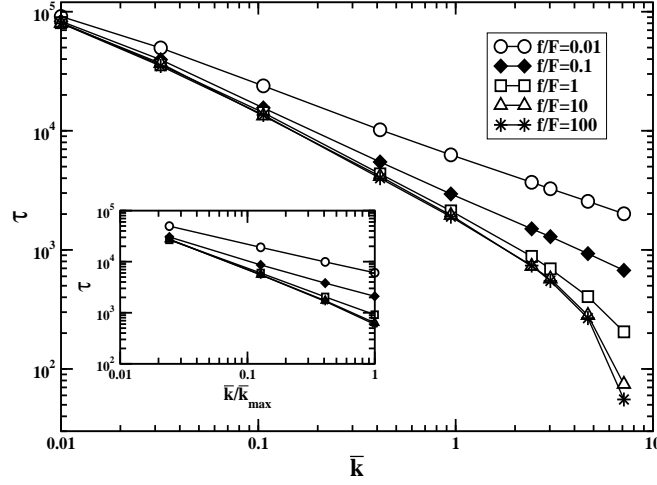
FIG. 1: Mean traversal time τ as a function of the mean shortcut degree $\bar{k}$ for different hopping rates ratios in the range $10^{-2} \leq f/F \leq 10^{2}$. These results correspond to hybrid SF/RL networks of size $N = 1000$, where the SF component has exponent $\gamma = 2$ and minimum degree $k_0 = 3$ (main plot) and $\gamma = 4$, $k_0 = 1$ (inset).

the traversal times considered in Ref. [11] to more general access times, considered as functions of the initial and final node degrees, and discuss further the role played by hubs in this context.

The rest of the paper is laid out as follows. In Section II, we introduce the basic model, describe the algorithm that we use to generate the hybrid networks that form the subject of this investigation, and set out our approach to studying the dynamics. In Section III, we present our results and a discussion. Section IV consists of concluding remarks.

II.   MODEL AND APPROACH

As stated in Section I, the aim of the present work is to explore the dynamics of random walks on hybrid network structures formed by superposing shortcut connections onto regular lattices. As in Ref. [11], the random walks considered here occur in continuous time but, unlike in that previous work, our interest here is in superposing *scale-free* structures. We thus provide below an explicit description first of how we build the networks and then of how we investigate the dynamics.

Each network of interest to the present study begins with a 1D lattice of N nodes (or sites) formed into a ring. Each node on this lattice is connected to each of its two immediate neighbors by a "regular" bond (or connection). On top of this translationally invariant substrate we build a hybrid network by algorithmically connecting certain pairs of nodes with "shortcut" connections. With shortcuts drawn using a power-law degree distribution, $P(k) \sim k^{-\gamma}$, a scale-free network of shortcuts can be superimposed on the underlying ordered ring, thus forming a hybrid scale-free/regular-lattice (SF/RL) network. Besides the exponent γ, the degree distribution of a finite SF network is defined by the minimum degree k_0 and the maximum (or cutoff) degree k_{cut} of the nodes participating in the network. Thus, in a normal "stand alone" scale free network, the number of sites with degree k is

$$N_k^{sf} = \begin{cases} (k/k_{cut})^{-\gamma} & \text{for } k_0 \leq k \leq k_{cut} \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

the total number of nodes is

$$N^{sf} = \sum_{k=k_0}^{k_{cut}} N_k^{sf}, \tag{2}$$

and the total degree is
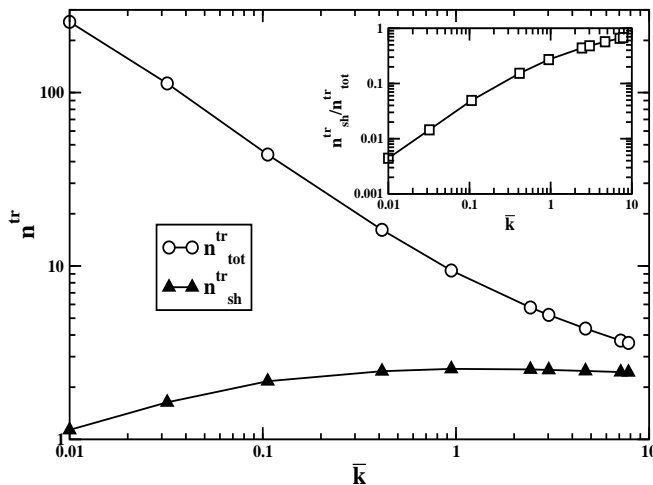
$$K = \sum_{k=k_0}^{k_{cut}} k N_k^{sf}. \tag{3}$$

FIG. 2: Average shortest traversal path as a function of the mean shortcut degree $\bar{k}$, where n_{tot}^{tr} refers to the total number of steps, while n_{sh}^{tr} is the number of steps taken across shortcut connections. Results correspond to hybrid SF/RL networks of size $N = 1000$, where the SF component has exponent $\gamma = 2$ and minimum degree $k_0 = 3$. The inset shows the ratio n_{sh}^{tr}/n_{tot}^{tr} vs $\bar{k}$ for the same networks.

In embedding a SF network of *shortcuts* into an already existing regular lattice of N nodes, we find that the SF network parameters are restricted by the requirement that the number of nodes interconnected by shortcuts be less than or equal to the total number of nodes on the lattice, i.e., $N^{sf} \leq N$. For a given lattice size N, this puts an upper bound on the value of the average shortcut degree $\bar{k} = K/N$ that can be realized. For the range of parameters and systems sizes studied here, the maximum average shortcut degree was bounded from above by values in the range $1 < \bar{k}_{max} < 10$.

A finite SF network may be characterized either through the original parameters γ, k_0, and k_{cut}, or through the parameters γ, k_0, and $\bar{k}$, replacing k_{cut} by $\bar{k}$ through the above relations. Our interest here being in comparing complex networks with the same average shortcut degree $\bar{k}$, we employ the second parameter triad $(\gamma, k_0, \bar{k})$ for network characterization.

In keeping with these ideas, to generate a particular realization of a hybrid SF/RL network with a given set of parameters $(\gamma, k_0, \bar{k})$, we randomly choose the lattice positions of the N^{sf} nodes that will belong to the SF shortcut subnetwork, and assign to each a given shortcut degree k following the power-law degree distribution (1). To each of the N_k^{sf} sites assigned shortcut degree k we associate k shortcut link ends, and then randomly connect pairwise the K link ends to establish the shortcut network (see e.g. [21]). This procedure, which is based on the so-called configuration model (see e.g. [15, 22, 23]) but includes an additional restriction on the maximum possible degree of the vertices, was shown to generate scale-free networks with no two- and three-vertex correlations [24, 25].

We now describe our approach for the analysis of random walks on hybrid SF/RL networks generated as explained above. Our interest being in continuous time random walks, as in Ref. [11], we could focus on the Master equation, which contains information about the network structure and the relevant hopping rates, numerically determine the set of propagators, or Green's functions, and obtain from them the desired transport quantities, following the method of Ref. [11]. Rather than pursue such a procedure, in the present paper we adopt for computational convenience a simpler Monte Carlo approach.

In the Monte Carlo calculations presented below, jump destinations of the random walker are chosen at each time step from local transition probabilities at the site occupied by the walker, by means of pseudo-random variables (see e.g. [26]). In particular, with transition rates F associated with jumps to neighboring sites on the ordered ring, and transition rates f describing jumps between pairs of sites connected by a shortcut, a walker located at a given site on the network with shortcut degree k will make a transition to one of its two neighbors on the ordered ring with probability $p_{lat} = FT$, and will make a transition to one of the k sites to which it is connected by a shortcut with probability $p_{sh} = fT$, where T is the duration of one Monte Carlo time step in the underlying continuous time random walk. The probability for the random walker to stay at its present position is thus $p_{stay} = 1 - (2F + kf)T$. After performing n_{MC} Monte Carlo steps, the corresponding continuous time elapsed is $\tau = Tn_{MC}$. The choice of T is arbitrary to the extent of keeping all transition probabilities positive definite (i.e. $T \leq (2F + k_{cut}f)^{-1}$). We have explicitly verified that this method accurately reproduces the results of the Master equation approach when applied to the SWN systems of Ref. [11].
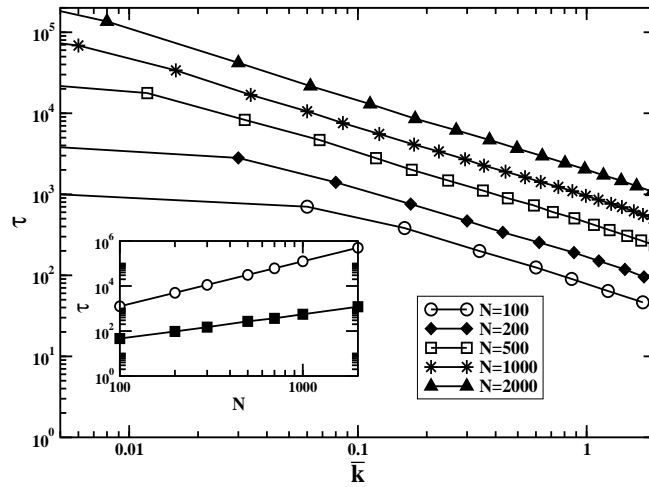
FIG. 3: Mean traversal time τ as a function of the mean shortcut degree $\bar{k}$ for networks of different size N, as indicated. The inset shows the mean traversal time as a function of network size N for two systems, one with $\bar{k} = 0$ (open circles), which scales as N^2 and one with $\bar{k} = 1.8$ (filled squares), which is very strongly connected, and for which the traversal time scales as N. In both we have taken $f/F = 1$.

As in Ref. [11], the focus on the present work is on the mean time for a walker to traverse the system, and on more generally defined *access* times, both of which are more straightforward to calculate using the Monte Carlo procedure of the present paper than the Master equation. Indeed, for each random walk trajectory, the traversal time τ is simply the elapsed time between the moment the walk starts, and the moment it arrives for the first time at the point $N/2$ sites away from which it started, measured in either direction around the ring. More generally, we can define the access time $\tau_{m,n}$ as the corresponding time for a walker starting at site m to arrive for the first time at site n. The mean access time, for a given network configuration, is then the average of this quantity over random walks starting from the same point on the same network, and then over the ensemble of networks characterized by the same set of network parameters. Except where explicitly noted, the numerical results shown in the present paper were obtained for networks of size $N = 1000$, typically averaged over 100 different network configurations for each set of network parameters, and over 1000 different random walk trajectories per configuration.

III. RESULTS AND DISCUSSION

Figure 1 shows the mean traversal time τ as a function of the mean shortcut degree $\bar{k}$ for different hopping rates ratios in the range $10^{-2} \leq f/F \leq 10^2$, as indicated. Here and throughout, traversal times are measured in units of F^{-1}, which is the timescale for jumps along regular lattice bonds. The results shown in the main plot correspond to hybrid SF/RL networks in which the SF component has exponent $\gamma = 2$ and minimum degree $k_0 = 3$, while the inset corresponds to $\gamma = 4$ and $k_0 = 1$. Notice that the adoption of a particular SF degree distribution, with given values of the parameters γ, k_0 and k_{cut}, defines the total number of shortcuts, and hence the value of the mean shortcut degree $\bar{k} = K/N$. This clearly differs from the SWN case, in which the mean shortcut degree is a free parameter. In order to generate plots of traversal times as functions of the shortcut density, we fix both γ and k_0, and consider different values of k_{cut} under the condition $N^{sf} \leq N$.

Figure 1 shows that the mean traversal time decreases monotonically as the mean shortcut degree increases, since a larger density of shortcuts naturally contributes to shorten the time needed to random walk across the network. The effect of decreasing the traversal times is only modest for small values of the rates ratio f/F, but is increasingly large for fast shortcuts ($f/F \gg 1$) and large shortcut densities close to the maximum degree ($\bar{k} \simeq \bar{k}_{max}$).

In the limit $\bar{k} \ll 1$, the shortcut density is very low and the traversal time tends to the diffusive limit $\tau_{diff} = (1/2F) \times (N/2)^2$. Even for $f/F \gg 1$, the fast shortcut connections are sparse and do not lead to a substantial reduction of the traversal time. This explains the data collapse onto a universal curve. However, when the number of shortcuts is increased above a threshold close to $\bar{k} \simeq \bar{k}_{max}$, which is related to the percolation of the shortcut network component of the hybrid structure, the transport mode changes from being dominated by diffusion on the lattice to being mainly due to propagation along fast shortcut connections. As pointed out above, this behavior is qualitatively similar to the mean traversal times observed in SWN and other network structures [11].
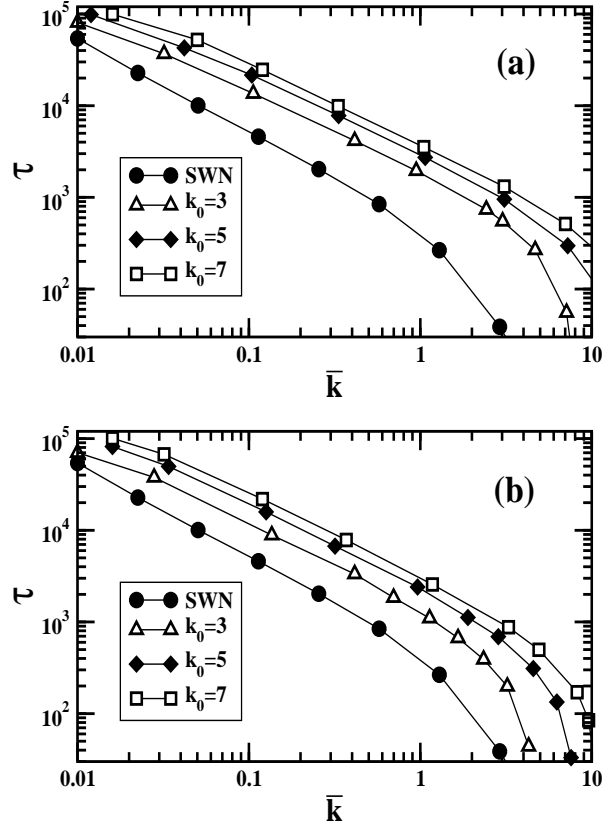
FIG. 4: Mean traversal time as a function of the mean shortcut degree, as obtained in the fast shortcut regime ($f/F = 100$) for networks of size $N = 1000$. The traversal times for hybrid SF/RL networks with different values of minimum degree ($k_0 = 3, 5$, and 7) are compared to corresponding SWN results. The values used for the exponent of the SF degree distribution are: (a) $\gamma = 2$ and (b) $\gamma = 3$.

The condition for having a percolation threshold in a complex network is that the degree distribution $P(k)$ satisfies

$$\sum_k k(k-2)P(k) > 0 \ , \tag{4}$$

which is known as the Molloy-Reed criterion [15]. This equation implies that the giant connected component is present for scale-free networks with $k_0 > 1$, irrespective of γ. However, if $k_0 = 1$, a finite percolation threshold exists only if $\gamma < 3.479$ [16]. The inset in Figure 1 shows traversal times corresponding to $\gamma = 4$ and $k_0 = 1$, where the data are indeed observed to collapse onto a universal curve for $f/F \gg 1$, even for $\bar{k} \approx \bar{k}_{max}$. Since the degree distribution is very steep, however, in this case we are restricted to $k_{cut} \leq 5$: only a few data points can be calculated.

Figure 2 shows the average shortest traversal path as a function of the mean shortcut degree $\bar{k}$, for hybrid SF/RL networks of size $N = 1000$, where the SF component has exponent $\gamma = 2$ and minimum degree $k_0 = 3$. We define n_{tot}^{tr} as the total number of steps involved in the shortest path for traversing the system, averaged over different initial sites and different network configurations. Analogously, we can define n_{sh}^{tr} (n_{lat}^{tr}) as the number of steps across shortcut connections (lattice bonds) corresponding to the same average shortest paths (such that $n_{tot}^{tr} = n_{sh}^{tr} + n_{lat}^{tr}$). The ratio n_{sh}^{tr}/n_{tot}^{tr} is shown in the inset in Figure 2 as a function of $\bar{k}$. This plot makes evident, particularly for large shortcut densities, the increasing importance of shortcut connections for traversing the system optimally.

Figure 3 shows how the mean traversal time τ as a function of $\bar{k}$ scales with the size of the network. We display in the main figure τ vs $\bar{k}$ plots for various values of the network size N. We display in the inset the mean traversal time as a function of network size N for two specific systems, one with $\bar{k} = 0$ (open circles), which scales as N^2 appropriate to the diffusive limit, and one with $\bar{k} = 1.8$ (filled squares), which is very strongly connected, and for which the traversal time scales as N. In both we have taken $f/F = 1$.

It is certainly of great interest to examine and possibly identify correlations between structural network features and their relative efficiency in signal transmission. With this aim, let us here compare quantitatively the SWN case to hybrid networks with different SF components.
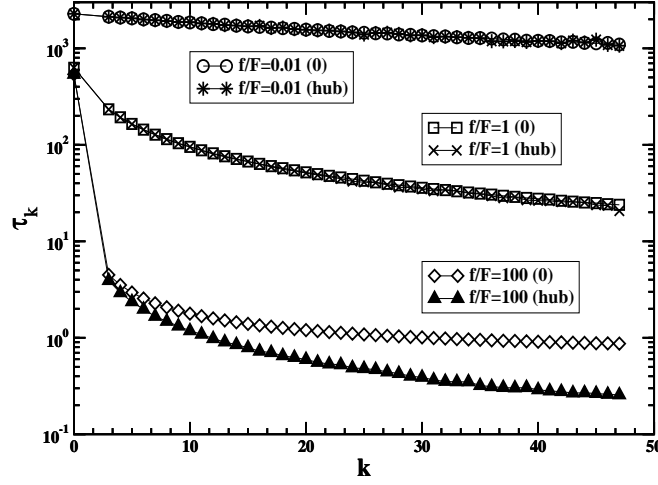
FIG. 5: Mean access times τ_k^0 and τ_k^{hub} as functions of the final node degrees, for hybrid SF/RL networks with $\gamma = 2$, $k_0 = 3$, $k_{cut} = 49$ and different hopping rates ratios, as indicated. See more details in the text.

Figure 4 shows a comparison of traversal times calculated for SWNs, which were studied in Ref. [11], with our new results for hybrid SF/RL networks with different values of the minimum degree ($k_0 = 3, 5$, and 7) and two different exponent values: (a) $\gamma = 2$ and (b) $\gamma = 3$. In this figure, all networks have the same total number of nodes, and are plotted as a function of the mean shortcut degree $\bar{k}$, which for fixed N is a direct measure of the total number of shortcuts $N_s = N\bar{k}/2$ in the network. Thus, by comparing traversal times of different networks of the same size at a given value of $\bar{k}$ we are able to address the question of how best to optimize the traversal time with a fixed number of shortcuts. The results displayed in this figure all correspond to $f/F = 100$, which is the relevant limit in the problem of network optimization through the addition of fast shortcuts.

The plots clearly reveal that, for a given value of the shortcut density, the SWNs yield generally lower traversal times. Hence, we conclude that inhomogeneously distributed shortcuts yield larger mean traversal times, when compared to the case of networks with homogeneously distributed shortcuts. Moreover, the traversal times for hybrid SF/RL networks show a monotonic dependence with their corresponding minimum degree: the larger k_0 (and thus, the more concentrated the shortcuts in a relatively small number of highly connected nodes), the larger the mean traversal times. These observations clearly indicate the key role played by the degree of inhomogeneity of the shortcut distribution in the network's transport properties.

Further insight into the role of shortcut distribution inhomogeneities can be gained by studying mean *access times* as functions of initial and final node degrees. We can define the mean access time $\tau_{i,k}$ as the average first passage time for going from an arbitrary initial site with degree i to an arbitrary final site with degree k. In particular, let us consider access times $\tau_k^0 \equiv \tau_{0,k}$ for initial sites without shortcut connections (i.e. sites that belong to the backbone regular lattice, but do not participate in the SF component), and $\tau_k^{hub} \equiv \tau_{k_{cut},k}$ for the case in which the initial site is the SF hub (i.e. the network's most connected site).

Figure 5 shows mean access times τ_k^0 and τ_k^{hub} as functions of the final node degrees, for hybrid SF/RL networks with $\gamma = 2$, $k_0 = 3$, $k_{cut} = 49$ and different hopping rates ratios in the range $10^{-2} \leq f/F \leq 10^2$. Notice that, for these network parameters, the resulting SF shortcut structure is quite dense ($N^{sf}/N = 0.9$ and $\bar{k} = 7.1$). For $f/F \leq 1$, it is observed that $\tau_k^0 \simeq \tau_k^{hub}$ irrespective of the degree of the final node. Indeed, this is the expected result for the case of relatively slow shortcuts, where the connections of the complex network component play a minor role. However, for $f/F \gg 1$, it is seen that $\tau_k^0 > \tau_k^{hub}$ for all $k \geq k_0$, which can be explained from the fact that walkers starting from isolated nodes need to take some steps along slow lattice bonds before reaching sites belonging to the fast shortcut SF component. Arguments along the same lines also explain the $k-$dependence of these plots, and particularly the large gaps observed in the $f/F \gg 1$ case between final nodes with degrees $k = 0$ and $k \geq k_0$.

Finally, let us consider the minimum, mean and maximum access times, and compare their behavior for different hopping rates ratios. For random walks starting at sites without shortcut connections, the minimum, mean and maximum access times are respectively defined as $\tau_{min}^0 = \min \tau_k^0$, $\bar{\tau}^0 = (k_{cut} - k_0 + 1)^{-1} \sum_k \tau_k^0$ and $\tau_{max}^0 = \max\{\tau_k^0\}$, with analogous definitions holding for walks starting at the SF hub.

Figure 6 shows these three access times as functions of the rates ratio f/F, for hybrid SF/RL networks with parameters $\gamma = 2$, $k_0 = 3$, $k_{cut} = 49$. As expected from the previous discussion, little difference is seen in the case of slow shortcuts, since then the transport mode is dominated by diffusion along lattice bonds. However, in the $f/F \gg 1$ regime, the minimum access times are found to differ substantially due to the existence of fast long-range shortcut
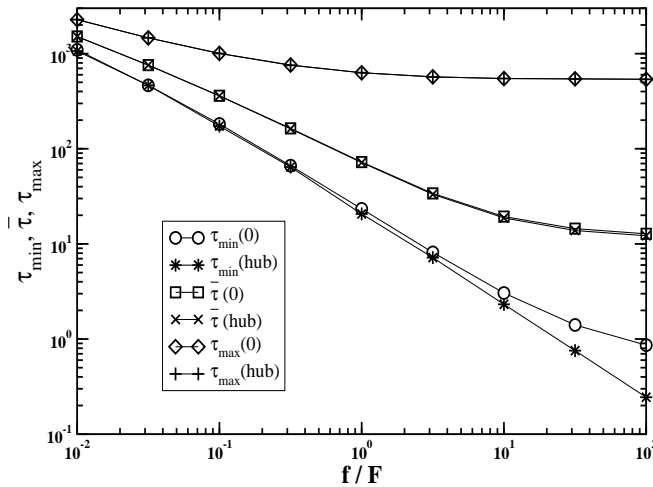
FIG. 6: Minimum, mean and maximum access times for random walks starting at sites without shortcut connections, as well as for walks starting at the SF hub. The hybrid SF/RL network parameters are $\gamma = 2$, $k_0 = 3$, $k_{cut} = 49$. See text for more details.

connections among highly connected nodes.

IV. CONCLUDING REMARKS

In summary, we have studied mean traversal and access times for random walks on hybrid scale-free/regular-lattice networks, and have considered different hopping rates for steps across lattice bonds and across network shortcuts. We have found two interesting results. First, in the limit of fast and sparse shortcut connections, traversal times collapse onto a universal curve. Second, a crossover behavior occurs related to the percolation threshold of the scale-free network component for higher shortcut densities. We have discussed the occurrence of the transition in terms of the Molloy-Reed criterion, which specifies the conditions for the existence of the giant connected component in the scale-free network structure. Although the qualitative behavior of traversal times is similar to observations recently reported in Newman-Watts small-world networks [11], the quantitative comparison reveals that mean traversal times are larger for networks with a higher degree of inhomogeneity in their link probability distribution. Scale-free networks represent the paradigm for complex random structures possessing inhomogeneous probability distribution of links. Our purpose in comparing SF network results with small world network results has been to study the effect of inhomogeneities in the degree distribution of links. Finally, by considering access times as functions of the degree of the initial and final sites, we have stressed the key role played by hubs in reducing the networks minimum access times when fast shortcuts are considered.

Acknowledgments

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